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Phase diagrams for two-dimensional six- and eight-states spin systems

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Abstract. We consider natural extensions of the Ashkin–Teller model to six- and eight-states spin systems. The phase diagrams corresponding to different symmetries are obtained using mean-field and Monte Carlo analysis combined with finite-size scaling. The critical indices corresponding to two new universality classes are obtained.

1. Introduction

In this paper we present a detailed study of the phase diagrams and critical indices for six- and eight-states systems in two dimensions. We have restricted our study to those systems which are essentially extensions of the Ashkin–Teller model (Ashkin and Teller 1943) which is well understood (Ditzian *et al* 1980).

For each system we have first done a mean-field calculation and then performed Monte Carlo analysis combined with finite-size scaling. For each point on the critical line describing a continuous transition we have computed the specific heats and susceptibilities determining α/ν and γ/ν.

We now describe the content of the paper.

In § 2 we define the systems and exhibit their symmetries. We also make the connection with previously studied models.

The Monte Carlo and finite-size scaling analysis (Fisher 1971) is described in § 3. We also give a subtraction method which improves the convergence of the estimates. The method is tested in the four-states Potts model (Potts 1952).

The phase diagrams for the six-states systems with $Z_2 \wr S_3$ and $S_3 \wr Z_2$ symmetries are presented in § 4. (When a model is invariant under a certain finite group G we label the system with G. For example a system with symmetry $Z_2 \wr S_3$ will be called a $Z_2 \wr S_3$ model.)

Sections 5–7 deal with eight-states models. Section 5 with the $Z_2 \wr Z_2 \wr Z_2$ model, § 6 with the $Z_2 \otimes S_4$ model and § 7 with the $Z_2 \wr S_4$ and $S_4 \wr Z_2$ models.

In § 8 we discuss our results and give the critical indices corresponding to the universality classes $Z_2 \wr S_3$ and $Z_2 \wr Z_2 \wr Z_2$.

2. The six- and eight-states Hamiltonian and their symmetries

We start with the six-states systems. The most general Hamiltonian which has at least

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$Z_6 = Z_2 \otimes Z_3$ symmetry is

$$\begin{aligned}
-H = \sum_{\langle ij \rangle} & (a_{00} + a_{10}(-1)^{(\alpha_i - \alpha_j)} + a_{01} \exp[\tfrac{2}{3}\pi i(\beta_i - \beta_j)] \\
& + a_{02} \exp[\tfrac{4}{3}\pi i(\beta_i - \beta_j)] + a_{11}(-1)^{(\alpha_i - \alpha_j)} \exp[\tfrac{2}{3}\pi i(\beta_i - \beta_j)] \\
& + a_{12}(-1)^{(\alpha_i - \alpha_j)} \exp[\tfrac{4}{3}\pi i(\beta_i - \beta_j)]) \\
= \sum_{\langle ij \rangle} & L(\alpha_i - \alpha_j, \beta_i - \beta_j)
\end{aligned} \tag{2.1}$$

where the sum covers all nearest neighbours on the lattice and

$$L(\alpha, \beta) = \sum_{\bar{\alpha}=0}^{1} \sum_{\bar{\beta}=0}^{2} a_{\bar{\alpha}\bar{\beta}}(-1)^{\bar{\alpha}\alpha} \exp(\tfrac{2}{3}\pi i \bar{\beta}\beta). \tag{2.2}$$

We have considered only ferromagnetic interactions.

In (2.1), α_i and β_i take values in Z_2 and respectively Z_3:

$$\alpha_i = 0 \text{ and } 1; \qquad \beta_i = 0, 1 \text{ and } 2. \tag{2.3}$$

The classification of higher symmetries of the Hamiltonian (2.1) which appear when some of the coupling constants $a_{\bar{\alpha}\bar{\beta}}$ are equal, was done by Marcu *et al* (1981) and is shown in table 1. In this table Z_N and S_N denote the cyclic and permutation group respectively of N objects. $G \wr H$ represents the wreath product of the groups G and H and $G \otimes H$ their direct product.

Table 1. Relations between the coupling constants for higher symmetries in the six-states spin model.

Global symmetry	Order of the group	Relations between the coupling constants
$Z_2 \otimes Z_3$	6	—
$Z_2 \otimes S_3$	12	$a_{0,1} = a_{0,2}$, $a_{1,1} = a_{1,2}$
$Z_3 \wr Z_2$	18	$a_{0,1} = a_{1,1}$, $a_{0,2} = a_{1,2}$
$Z_2 \wr Z_3$	24	$a_{1,0} = a_{1,1} = a_{1,2}$
$Z_2 \wr S_3$	48	$a_{0,1} = a_{0,2}$, $a_{1,0} = a_{1,1} = a_{1,2}$
$S_3 \wr Z_2$	72	$a_{0,1} = a_{0,2} = a_{1,1} = a_{1,2}$
S_6	720	$a_{1,0} = a_{0,1} = a_{0,2} = a_{1,1} = a_{1,2}$

Out of the seven models described in table 1, two have already been studied in detail. Those are the $Z_2 \otimes S_3$ model and the S_6 model. The first one corresponds to the vector Potts model (Elitzur *et al* 1979) where one takes $a_{01} = a_{02} = a_{10} = 0$ or to the Domany-Riedel model (Domany and Riedel 1979) (when $a_{01} = a_{02} \neq 0$, $a_{10} \neq 0$).

We would like to understand the phase diagrams of all the models in table 1. In this paper we will focus however only on the models with $Z_2 \wr S_3$ and $S_3 \wr Z_2$ symmetries. The models with $Z_3 \wr Z_2$ and $Z_2 \wr Z_3$ symmetries do not conserve parity and similar to the asymmetric clock model (Ostlund 1981, Huse 1981) can eventually present incommensurable structures.

Using the identities:

$$\begin{aligned}
&1 + (-1)^{\alpha} = 2\delta(\alpha) && (\alpha = 0, 1) \\
&1 + \exp(\tfrac{2}{3}\pi i\beta) + \exp(\tfrac{4}{3}\pi i\beta) = 3\delta(\beta) && (\beta = 0, 1, 2)
\end{aligned} \tag{2.4}$$

where $\delta(\alpha)$, $\delta(\beta)$ are the Kronecker functions, we will write the two models in a slightly different form:

$$L=3J[(1-2z)+2z\delta(\alpha)]\delta(\beta) \tag{2.5}$$

for the $Z_2 \wr S_3$ model and

$$L=2\tilde{J}[(1-2\tilde{z})+3\tilde{z}\delta(\beta)]\delta(\alpha) \tag{2.6}$$

for the $S_3 \wr Z_2$ model.

The two models given by (2.5) and (2.6) are special cases of the Domany-Riedel model. The $Z_2 \wr S_3$ model is also known as the $N=3$ discrete spin cubic model (Aharony 1977).

We now consider eight-states models which have at least $Z_2 \otimes Z_2 \otimes Z_2$ symmetry. Those are the natural extensions of the four-states Ashkin-Teller models which have at least $Z_2 \otimes Z_2$ symmetry.

The Hamiltonian is

$$-H=\sum_{\langle i,j\rangle} L(\alpha_i^{(1)}-\alpha_j^{(1)}, \alpha_i^{(2)}-\alpha_j^{(2)}, \alpha_i^{(3)}-\alpha_j^{(3)}) \tag{2.7}$$

where

$$L(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})=\sum_{\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3=0}^{1} a_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3}(-1)^{\sum_{\mu=1}^{3}\bar{\alpha}_\mu\alpha^{(\mu)}} \tag{2.8}$$

and $\alpha_i^{(\mu)}$ take values in Z_2:

$$\alpha_i^{(\mu)}=0 \text{ and } 1. \tag{2.9}$$

The different symmetries of this Hamiltonian which occur for special choices of the coupling constants $a_{\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3}$ have been found by Marcu *et al* (1981), and are displayed in table 2. In this paper we concentrate only on systems with higher symmetries starting with the $Z_2 \wr Z_2 \wr Z_2$ model. We take arbitrarily $a_{111}=0$ and define our model (see table 2):

$$L=J[x((-1)^{\alpha^{(3)}}+(-1)^{\alpha^{(1)}+\alpha^{(2)}})+(1-x)((-1)^{\alpha^{(1)}}+(-1)^{\alpha^{(2)}})(1+(-1)^{\alpha^{(3)}})]. \tag{2.10}$$

This model is new and defines a universality class.

Table 2. Relations between the coupling constants for higher symmetries in the eight-states spin model.

Global symmetry	Order of the group	Relations between the coupling constants
$Z_2 \otimes Z_2 \otimes Z_2$	8	—
$Z_2 \otimes (Z_2 \wr Z_2)$	16	$a_{100}=a_{011}$, $a_{010}=a_{101}$
$Z_2 \wr (Z_2 \otimes Z_2)$	64	$a_{100}=a_{010}=a_{101}=a_{011}$
$(Z_2 \otimes Z_2) \wr Z_2$	32	$a_{100}=a_{011}$, $a_{010}=a_{101}$, $a_{001}=a_{110}$
$Z_2 \wr Z_2 \wr Z_2$	128	$a_{100}=a_{010}=a_{101}=a_{011}$, $a_{001}=a_{110}$
$Z_2 \otimes S_4$	48	$a_{100}=a_{010}=a_{001}$, $a_{110}=a_{101}=a_{011}$
$Z_2 \wr S_4$	384	$a_{100}=a_{010}=a_{101}=a_{011}$, $a_{110}=a_{001}=a_{111}$
$S_4 \wr Z_2$	1 152	$a_{100}=a_{010}=a_{001}=a_{110}=a_{101}=a_{011}$
S_8	40 320	$a_{100}=a_{010}=a_{001}=a_{110}=a_{101}=a_{011}=a_{111}$

For the $Z_2 \otimes S_4$ model, we take again arbitrarily $a_{111} = 0$ and obtain

$$L = J[y((-1)^{\alpha^{(1)}+\alpha^{(2)}} + (-1)^{\alpha^{(1)}+\alpha^{(3)}} + (-1)^{\alpha^{(2)}+\alpha^{(3)}}) + (1-y)((-1)^{\alpha^{(1)}} + (-1)^{\alpha^{(2)}} + (-1)^{\alpha^{(3)}})] \quad (2.11)$$

The model described by (2.11) has already been considered by Grest and Widom (1981).

The $Z_2 \wr S_4$ and $S_4 \wr Z_2$ models are given by two coupling constants only. Using the identities (2.4) we can bring them to the form:

$$L = 4J[(1-2z) + 2z\delta(\alpha^{(1)})]\delta(\alpha^{(2)})\delta(\alpha^{(3)}) \quad (2.12)$$

in the $Z_2 \wr S_4$ case and

$$L = 2\tilde{J}[(1-2\tilde{z}) + 4\tilde{z}\delta(\alpha^{(2)})\delta(\alpha^{(3)})]\delta(\alpha^{(1)}) \quad (2.13)$$

for the $S_4 \wr Z_2$ model.

The $Z_2 \wr S_4$ model coincides with the $N = 4$ discrete cubic one (Aharony 1977).

The S_8 model is the eight-states Potts model.

In the next sections we will present a systematic study of the systems described by (2.5), (2.6), and (2.10)-(2.13). First we describe our methods of analysis.

3. Determination of the phase diagrams and critical indices

For each model we have done both mean-field calculations and a Monte Carlo analysis combined with finite-size scaling.

The mean-field calculations were done using standard methods (see, for example, Grest and Widom 1981).

In our Monte Carlo analysis we have considered $n \times n$ lattices and measured the specific heat and susceptibility using the importance-sampling Metropolis method (Metropolis *et al* 1953). We have then performed a finite-size scaling analysis (Fisher 1971). Getting estimates for the critical points $T_{c,n}$ and the ratio of the critical exponents $(\alpha/\nu)_n(\gamma/\nu)_n$. In order to improve our convergence we have used a method inspired by the way resonances are separated from the background in particle physics. We first notice that in the thermodynamic limit, the specific heat (identical considerations can be used for the susceptibility) has a regular part and a divergent part:

$$C(T) = C_{\text{reg}}(T) + C_{\text{div}}(T). \quad (3.1)$$

On a $n \times n$ lattice, the specific heat $C_n(T)$ is a smooth function with a bump. We now notice that outside the critical region (where the correlation length is finite) $C_n(T)$ converges exponentially to $C_{\text{reg}}(T)$. In practice this implies that if we are not too close to the critical point T_c we can fit $C_n(T)$

$$C_n(T) = C_{\text{reg}}(T) + A(T) \cdot \exp(-n/\xi(T)) \quad (3.2)$$

and determine $C_{\text{reg}}(T)$. This can be done for temperatures lower and higher than T_c. We then interpolate by hand in order to get approximately $C_{\text{reg}}(T)$ over the whole region. The finite-size scaling analysis is now performed on the quantity

$$\bar{C}_n(T) = C_n(T) - C_{\text{reg}}(T). \quad (3.3)$$

The estimates for the critical point $T_{c,n}$ are obtained from the positions of the maxima of $\bar{C}_n(T)$ and the estimates for $(\alpha/\nu)_n$ come from the values of the maxima.

The values for T_c and α/ν have been obtained both through fits and Vanden Broeck-Schwartz approximants (Vanden Broeck and Schwartz 1979).

The method was checked by considering the four-states Potts model defined by the Hamiltonian:

$$-H = 4J \sum_{\langle i,j \rangle} \delta(\beta_i - \beta_j) \tag{3.4}$$

where $\beta = 0, 1, 2, 3$. We have taken lattices of size $n = 5, 7, 10, 14, 20, 28, 40$ and obtained $T_c = 3.64 \pm 0.01$, $\alpha = 0.65 \pm 0.03$ and $\eta = 0.26 \pm 0.04$ (throughout this paper the hyperscaling relations $\alpha = 2(1-\nu)$ and $\eta = 2 - \gamma/\nu$ are assumed). This is to be compared with the exact result $T_c = 3.6409\ldots$ and $\alpha = \frac{2}{3}$ and $\eta = \frac{1}{4}$ (Baxter 1982) and the Monte Carlo estimate of Swendsen *et al* (1981), $\alpha = 0.66 \pm 0.01$.

4. The $Z_2 \wr S_3$ and $S_3 \wr Z_2$ systems

We first notice the six-states models with symmetry $Z_2 \wr S_3$ and $S_3 \wr Z_2$ described by (2.5) and (2.6) are special cases ($N = 3$) of the more general $Z_2 \wr S_N$ models given by

$$L = NJ[(1-2z) + 2z\delta(\alpha)]\delta(\beta) \qquad (\alpha = 0, 1; \beta = 0, 1, \ldots, N-1) \tag{4.1}$$

and the $S_N \wr Z_2$ models given by

$$L = 2\tilde{J}[(1-2\tilde{z}) + N\tilde{z}\delta(\beta)]\delta(\alpha) \qquad (\alpha = 0, 1; \beta = 0, 1, \ldots, N-1). \tag{4.2}$$

Notice that the $Z_2 \wr S_4$ and $S_4 \wr Z_2$ models described by (2.12) and (2.13) correspond to the choice $N = 4$ in (4.1) and (4.2).

We observe that for the special points $z = \tilde{z} = \frac{1}{2}$ both models (4.1) and (4.2) become the $2N$-states Potts model with symmetry S_{2N}. For $z = 0$ one obtains the N-states model whereas for $\tilde{z} = 0$ one gets the Ising model.

In two dimensions the models described by (4.1) and (4.2) are related through a duality transformation

$$\begin{aligned}
&z = \tfrac{1}{2}(A-B)/(A-C), \qquad J/kT = (A-C)/N \\
&e^A = a + Nb + (N-1)c \\
&e^B = a - Nb + (N-1)c \\
&e^c = a - c \\
&a = \exp\{\tilde{J}[1 + (2N-3)\tilde{z}]/(kT)\} \\
&b = \exp[-\tilde{J}(1-\tilde{z})/(kT)] \\
&c = \exp[\tilde{J}(1-3\tilde{z})/(kT)].
\end{aligned} \tag{4.3}$$

We now specialise to the case $N = 3$. We have done mean-field calculations on both models and the results are displayed in figure 1. (In figures 1-4 the broken curves indicate first-order phase transitions and full curves second-order phase transitions.) The curves HFE and DF correspond to the $Z_2 \wr S_3$ model, the curves H′F′E′ and DF′ to the $S_3 \wr Z_2$ model. In this way one gets lower and upper estimates of the critical temperatures. The end point on the first-order line FE corresponding to $z = 1$ is special because there the transition is second order with $\alpha = \frac{1}{2}$ and $\gamma = 1$.

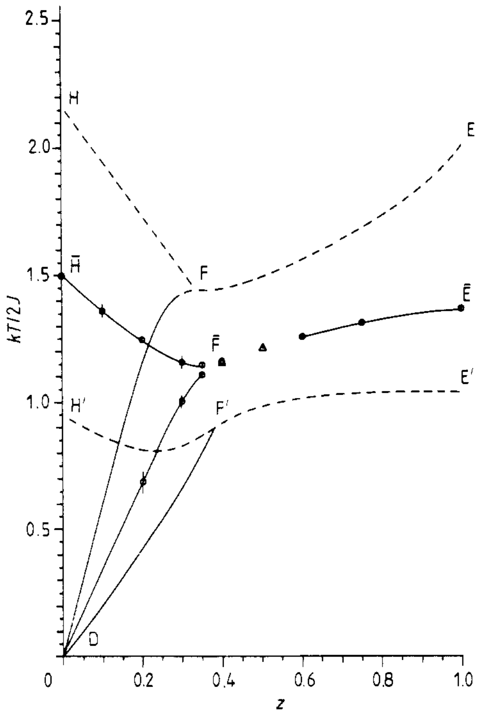

Figure 1. Phase diagram for the $Z_2 \wr S_3$ model. The lines HFE, FD are obtained from mean-field analysis. The line H′F′E′, F′D from the mean field applied to the dual model. The lines $\bar{H}\bar{F}\bar{E}$, $\bar{F}$D are obtained from the Monte Carlo analysis.

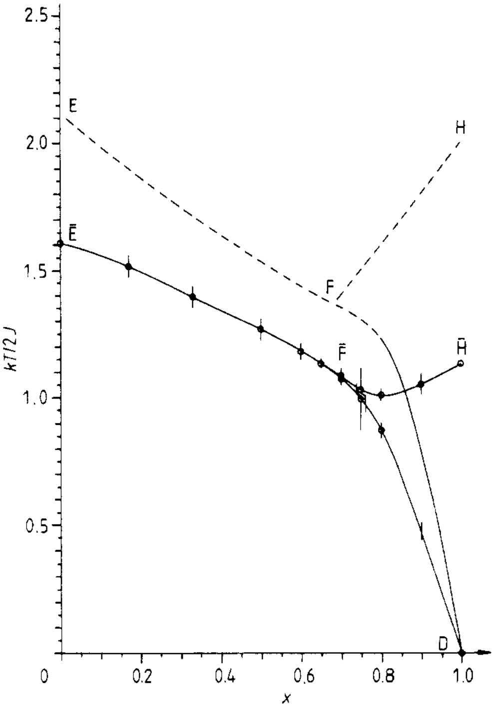

Figure 2. Phase diagram for the $Z_2 \wr Z_2 \wr Z_2$ model. The lines HFE, FD are obtained from mean-field analysis. The lines $\bar{H}\bar{F}\bar{E}$, $\bar{F}$D are obtained from the Monte Carlo analysis.

We will discuss the physical interpretation of the phase diagram using the lines obtained in the $Z_2 \wr S_3$ model. We have a high-temperature paramagnetic phase (limited by HFE), a low-temperature ferromagnetic phase (limited by DFE) the whole symmetry being broken and an intermediate phase (limited by HFD) where the $Z_2 \wr S_3$ symmetry is partially broken to Z_2. We thus expect the HF line to be in the three-states Potts universality class (with $\alpha = \frac{1}{3}$ and $\eta = \frac{4}{15}$) and the line DF to be in the Ising universality class (with $\alpha = 0$ and $\eta = \frac{1}{4}$). The FE line should have a new nature.

Our Monte Carlo results are given in table 3 and the critical temperatures are shown also in figure 1. The critical line $\bar{H}\bar{F}$ ($z_{\bar{F}} \approx 0.4$) is of Potts type (compare the indices with the exact values given for $z = 0$). For the lower critical line (D$\bar{F}$) we have determined only the critical temperatures. Let us now consider the line $\bar{F}\bar{E}$ ($0.4 < z < 1$) and separate it into two pieces. In the region ($0.4 < z < 0.6$) our measurements were not able to distinguish between a first-order transition with a small latent heat and a continuous transition with a large α. The special point $z = \frac{1}{2}$ is known analytically (Baxter 1982) to be of first order. In the second region ($0.6 \leq z \leq 1$) we have clearly a line of second-order transitions with apparently the same critical exponents $\alpha = 0.71$ and $\eta = 0.31$.

Let us point out that our measurements of the susceptibility from which we have obtained $\eta = 2 - \gamma/\nu$ were done only on the $Z_2 \wr S_3$ model and that we think that they should be repeated on the $S_3 \wr Z_2$ model. We see no reason why the exponents should be the same.

Table 3. Critical points and exponents for the $Z_2 \wr S_3$ model. The lattice sizes used for the Monte Carlo analysis were $n \times n$.

z	kT_c/J	α/ν	η	α	n
0	2.9849	0.4	$\frac{4}{15}$	$\frac{1}{3}$	exact (Potts model ($q=3$))
0.1	2.72 ± 0.02				28
0.2	2.49 ± 0.01	0.57 ± 0.11	0.29 ± 0.07	0.44 ± 0.07	10, 14, 20, 28
	1.37 ± 0.04		0.25 ± 0.15		10, 20, 28
0.3	2.31 ± 0.02	0.46 ± 0.25		0.37 ± 0.17	20, 28
	2.00 ± 0.02				20, 28
0.35	2.294 ± 0.005				28, 40
	2.21 ± 0.01				28, 40
0.4	2.320 ± 0.005				30, 40
0.5	2.4228	first order		1.0	exact (Potts model ($q=6$))
0.6	2.51 ± 0.01	1.18 ± 0.03	0.28 ± 0.03	0.74 ± 0.01	5, 7, 10, 14, 20, 28
0.75	2.624 ± 0.003	1.15 ± 0.05	0.32 ± 0.05	0.73 ± 0.02	5, 7, 10, 14, 20, 28
1.0	2.73 ± 0.01	0.99 ± 0.06	0.32 ± 0.08	0.66 ± 0.03	7, 10, 14, 20, 30

5. The $Z_2 \wr Z_2 \wr Z_2$ model

This model is described by (2.10). We notice that there are three special points. For $x=0$ we find using table 2 that the system has a higher symmetry: $Z_2 \wr S_4$ and for $x=\frac{1}{2}$ it has the symmetry $S_4 \wr Z_2$. In the point $x=1$ we have two decoupled Ising models.

The results of the mean-field calculations are shown in figure 2. We have a high-temperature paramagnetic phase (separated by the line EFH), a low-temperature ferromagnetic phase (separated by the line EFD) where the $Z_2 \wr Z_2 \wr Z_2$ symmetry is completely broken and an intermediate phase (separated by the line HFD) where the $Z_2 \wr Z_2 \wr Z_2$ symmetry is broken to Z_2. We expect the line FH to be of Ashkin–Teller type, the line FD to be of Ising type and the line EF should be of a new type. This picture is confirmed by our Monte Carlo analysis. The results are shown in table 4 and the critical points are displayed in figure 2.

The line $\bar{\mathrm{E}}\bar{\mathrm{F}}$ ($x_{\bar{\mathrm{F}}} \approx 0.6$) corresponds to a new universality class with

$$\alpha = 0.80 \pm 0.02, \qquad \eta = 0.33 \pm 0.05 \tag{5.1}$$

since we see no change of the critical exponents along this line. The $\bar{\mathrm{F}}\bar{\mathrm{H}}$ line is presumably of Ashkin–Teller type but this information is based essentially only on one point ($x=0.8$) since for the other points the errors are very large. Nevertheless for $x>0.8$ we can see that the critical index α is small.

6. The $Z_2 \otimes S_4$ model

This model is defined by (2.11). Again we have three special points in this model. For $y=0$ we get three decoupled Ising models. For $y=\frac{1}{2}$ the symmetry is $S_4 \wr Z_2$ and for $y=1$ we get a four-states Potts model. The mean-field results as well as the Monte Carlo points are shown in figure 3. The critical indices are given in table 5. Based on our measurment and on the observation that the point $y=\frac{1}{2}$ should have the same critical indices (see (5.1)) as the point $x=\frac{1}{2}$ in the $Z_2 \wr Z_2 \wr Z_2$ model one can come to the following interpretation of the phase diagram. We have a high-temperature

Table 4. Critical points and exponents for the $Z_2 \wr Z_2 \wr Z_2$ model.

x	kT_c/J	α/ν	η	α	n
0	3.22 ± 0.02	1.35 ± 0.06	0.33 ± 0.05	0.81 ± 0.02	5, 7, 10, 14, 17, 20, 22, 30 (η without 20)
0.167	3.03 ± 0.04				20
0.333	2.79 ± 0.04	1.30 ± 0.06	0.53 ± 0.21	0.79 ± 0.02	5, 7, 10, 14, 20, 30 (η without 5)
0.5	2.53 ± 0.04				20
0.6	2.35 ± 0.03	1.35 ± 0.07	0.09 ± 0.15	0.81 ± 0.02	5, 7, 10, 14, 20, 30 (η without 30)
0.65	2.26 ± 0.01				30
	2.25 ± 0.01				30
0.7	2.16 ± 0.03				40
	2.15 ± 0.02				40
0.75	2.05 ± 0.09				30
	1.98 ± 0.12				30
0.8	2.01 ± 0.02	0.57 ± 0.12	0.47 ± 0.15	0.44 ± 0.07	14, 20, 30, 40
	1.74 ± 0.03	$0 \leq \alpha/\nu \leq 0.1$	$0.8 \leq \eta \leq 2$		14, 20, 30, 40
0.9	2.1 ± 0.04	$0 \leq \alpha/\nu \leq 0.45$	$0.7 \leq \eta \leq 2$		14, 20, 30
	0.95 ± 0.03	$0 \leq \alpha/\nu \leq 0.35$	$1.1 \leq \eta \leq 2$		14, 20, 30
1.0	2.2692	0	0.25	0	exact (Ising model)

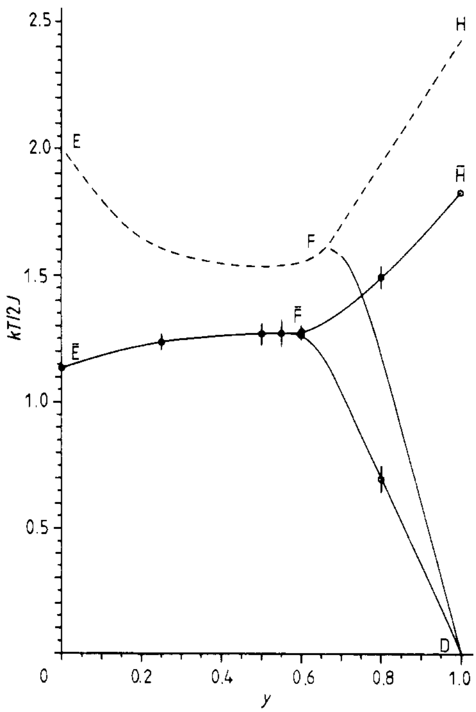

Figure 3. Phase diagram for the $Z_2 \otimes S_4$ model. The lines HFE, FD are obtained from mean-field analysis. The lines $\bar{H}\bar{F}\bar{E}$, $\bar{F}D$ from the Monte Carlo analysis.

Table 5. Critical points and exponents for the $Z_2 \otimes S_4$ model.

y	kT_c/J	α/ν	η	α	n
0	2.2692	0	0.25	0	exact (Ising model)
0.25	2.47 ± 0.03	0.93 ± 0.08	0.46 ± 0.05	0.633 ± 0.035	5, 7, 10, 14, 20, 30 (η without 30)
0.5	2.53 ± 0.04				20
0.55	2.54 ± 0.05				30
0.6	2.55 ± 0.03				30
	2.52 ± 0.02				30
0.8	2.98 ± 0.04	0.85 ± 0.03	0.25 ± 0.06	0.595 ± 0.015	5, 7, 10, 14, 20, 30
	1.39 ± 0.05	$0 \leqslant \alpha/\nu \leqslant 0.35$	0.54 ± 0.2		10, 14, 20, 30
1.0	3.641	1.0	0.25	$\frac{2}{3}$	exact (Potts model ($q=4$))

paramagnetic phase separated from a ferromagnetic phase by a line $\bar{\mathrm{E}}\bar{\mathrm{F}}$ ($y_{\bar{\mathrm{F}}} \approx \frac{1}{2}$) with exponents varying continuously ($\alpha = 0$ for $y = 0$, $\alpha = 0.8$ for $y = \frac{1}{2}$). The paramagnetic phase is separated from the intermediate phase by the line $\bar{\mathrm{F}}\bar{\mathrm{H}}$ which is of the four-states Potts type. Finally, the intermediate phase is separated from the ferromagnetic one by the line $\bar{\mathrm{F}}\bar{\mathrm{D}}$ which is Ising type. Our results are at variance with those of Grest and Widom (1981) who have considered the same model and used a Monte Carlo analysis only for a given lattice (no finite-size scaling). According to Grest and Widom the whole line $\bar{\mathrm{E}}\bar{\mathrm{F}}\bar{\mathrm{H}}$ should be first order.

7. The $Z_2 \wr S_4$ and $S_4 \wr Z_2$ models

The $Z_2 \wr S_4$ and $S_4 \wr Z_2$ models are described by (4.1) and (4.2) by writing $N = 4$. As discussed in § 4, in two dimensions the $S_4 \wr Z_2$ model is dual to the $Z_2 \wr S_4$ one. We have performed mean-field calculations for both models and the results are shown in figure 4. For two points the results are known exactly. At $z = 0$ we have the four-states Potts model with a continuous transition and at $z = \frac{1}{2}$ we have the eight-states Potts model with a discontinuous transition. The point $z = 1$ corresponds to the $y = \frac{1}{2}$ point of the $Z_2 \otimes S_4$ model so it is also known. We have not done a Monte Carlo analysis for the whole range of z although it should be done.

8. Conclusions

We have shown that generalisations of the Ashkin–Teller model to six and eight-states display phase diagrams with an interesting structure. Our study was limited to two dimensions. Some partial results in higher dimensions are given in the following paper (Badke *et al* 1984).

Two new universality classes have been found. One for the six-states model with symmetry $Z_2 \wr S_3$ where

$$\alpha = 0.71, \qquad \eta = 0.31 \tag{8.1}$$

and one for the eight-states model with symmetry $Z_2 \wr Z_2 \wr Z_2$ where

$$\alpha = 0.80, \qquad \eta = 0.33 \tag{8.2}$$

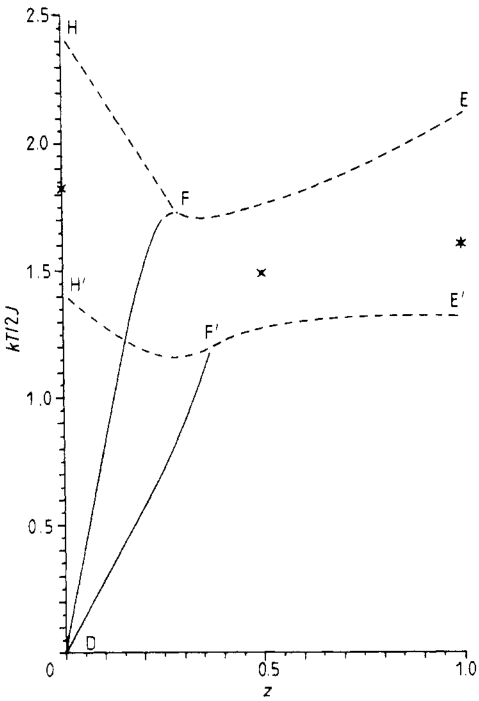

Figure 4. Phase diagram for the $Z_2 \wr S_4$ model. The lines HFE, FD are obtained from mean field analysis. The lines H'F'E', F'D from mean-field analysis for the dual model. The origin of values assigned to the points $z = 0$, $\frac{1}{2}$ and 1 are explained in the text.

For larger symmetries like $Z_2 \otimes S_4$ no new universality classes were found.

It will be interesting to see if one could obtain the indices (8.1) and (8.2) using the conformal algebra approach (Belavin *et al* 1984, Dotsenko and Fateev 1984).

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